Examples: Compute the following indefinite integrals.

- $\displaystyle\int \cos(x^2)dx.$

- $\displaystyle\int 2x\cos(x^2)dx.$

- $\displaystyle\int t^2 e^{5t^3}dt$

- $\displaystyle\int \cos x\sqrt{\sin x + 1}dx$

- $\displaystyle\int \frac{x}{1+x^2}dx.$

- $\displaystyle\int \frac{x}{1+x^4}dx.$

- $\displaystyle\int \sqrt{t^3 - 2t}(3t^2 - 2)dt.$

More Examples: Compute the following indefinite integrals.

- $\int \cos\theta e^{\sin\theta} d\theta.$

- $\int \frac{e^x + e^{-x}}{e^x - e^{-x}} dx.$

- $\int \frac{e^{\sqrt{y}}}{\sqrt{y}} dy.$

- $\int \cos(2x) dx$

- $\int e^{3x} dx$

- $\int (5x-1)^3 dx$

- $\int \frac{1}{2x} dx$

Method of Substitution

Let w be the "inside function" and $dw = \frac{dw}{dx}dx = w'(x)dx$.

This method "undoes" the chain rule. If we are trying to find

$$\int f(g(x))g'(x)dx,$$

substitution allows us to rewrite this integral in a simpler form. We let $w = g(x)$, (because $g(x)$ is the "inner function" of a composition), then

$$dw = \frac{dw}{dx}dx = g'(x)dx$$

and

$$\int f(g(x))g'(x)dx = \int f(w)dw.$$

Justification of Substitution

Let $F' = \dfrac{dF}{dx} = f$. Now

$$\frac{d}{dx}[F(g(x))] = f(g(x))g'(x),$$

thus

$$\int f(g(x))g'(x)dx = F(g(x)) + C.$$

If we let $w = g(x)$, then $\dfrac{dw}{dx} = g'(x)$, and

$$\int f(w)\frac{dw}{dx}dx = F(w) + C.$$

However, we also have

$$\int f(w)dw = F(w) + C.$$

Finally, we have

$$\int f(w)\frac{dw}{dx}dx = \int f(w)dw. \qquad \triangle$$

Examples: Calculate the following definite integrals:

- $\displaystyle\int_1^2 x\sin(x^2)dx$

- $\displaystyle\int_0^1 \frac{x^2}{1+x^3}dx$

- $\displaystyle\int_{\pi^2}^{9\pi^2/4} \frac{\cos\sqrt{x}}{\sqrt{x}}dx$

- $\displaystyle\int_0^2 \frac{t}{1+3t^2}dt$

- $\displaystyle\int_0^2 \frac{x}{(1+x^2)^2}dx$

- $\displaystyle\int_1^4 x\sqrt{x^2+4}dx$

- $\displaystyle\int_1^3 \frac{\sqrt{1+\frac{1}{x}}}{x^2}dx$

Examples: Calculate the following definite integrals:

- $\displaystyle\int_1^2 \sin(x^2)dx$

- $\displaystyle\int_0^1 \frac{x^2}{1+x^4}dx$

Example: Calculate the following indefinite integrals:

- $\displaystyle\int x\sqrt{x-1}dx$

- $\displaystyle\int x^3\sqrt{1+x^2}dx$

- $\displaystyle\int \sin^3\theta d\theta.$

- $\displaystyle\int \sqrt{1+\sqrt{x}}dx.$